Gamma matrix conventions in our code

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Start with the Dirac operator in Minkowski space

$$\not{D} - m = i\gamma^\mu \partial_\mu - m \quad (1)$$
$$= i\gamma^0 \frac{\partial}{\partial t} + i\gamma^i \frac{\partial}{\partial x^i} - m \quad (2)$$

Note carefully the use of the *covariant* partial derivative. Now, analytically continue to Euclidean space-time,

$$t \rightarrow -i\tau \quad (3)$$
$$\gamma_E^i = -i\gamma^i \quad (4)$$
$$\gamma_E^4 = \gamma^0 \quad (5)$$

The above definitions result because we must flip the sign of both time and space derivative terms to have the same sign as the mass term:

$$\not{D} - m = i\gamma^\mu \partial_\mu - m \quad (6)$$
$$\rightarrow i\gamma^4 \frac{\partial}{\partial(-i\tau)} + -(-i)\gamma^i \frac{\partial}{\partial x^i} - m \quad (7)$$
$$= -\gamma^4 \frac{\partial}{\partial \tau} - \gamma_E^i \frac{\partial}{\partial x^i} - m \quad (8)$$
$$= -\left(\gamma_E^\mu \partial_\mu + m\right). \quad (9)$$

In Minkowski space we define (this seems to be everyone's convention)

$$\gamma_5 = \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma_0\gamma_1\gamma_2\gamma_3 \quad (10)$$
$$\rightarrow i(i^3)(-i)^3\gamma^4\gamma^1\gamma^2\gamma^3 \quad (11)$$
$$= -\gamma_E^1\gamma_E^2\gamma_E^3\gamma_E^4 \quad (12)$$
$$\equiv -\gamma_E^5 \quad (13)$$

Again, note the signs for (co)contravariant gamma matrices, and the last minus sign comes from anti-commuting γ^4 to the end.

The chiral basis in Minkowski space is (*i.e.*, Peskin)

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

where $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Our code conventions (`alg_qpropw::Wilson_matrix`):

gamma(XUP)	gamma(YUP)	gamma(ZUP)	gamma(TUP)	gamma(FIVE)
0 0 0 i	0 0 0 -1	0 0 i 0	0 0 1 0	1 0 0 0
0 0 i 0	0 0 1 0	0 0 0 -i	0 0 0 1	0 1 0 0
0 -i 0 0	0 1 0 0	-i 0 0 0	1 0 0 0	0 0 -1 0
-i 0 0 0	-1 0 0 0	0 i 0 0	0 1 0 0	0 0 0 -1

Thus, it appears that our conventions are

$$\begin{aligned} i\gamma^1 &= \gamma_E^1 && (14) \\ -i\gamma^2 &= \gamma_E^2 && (15) \\ i\gamma^3 &= \gamma_E^3 && (16) \\ \gamma^0 &= \gamma_E^4 && (17) \end{aligned}$$

or that our $\gamma^{1,3}$ are minus everyone else's

I have checked in Mathematica (my Dirac.nb) that $\gamma_E^1\gamma_E^2\gamma_E^3\gamma_E^4 = \gamma_E^5$. Now, our $\gamma_E^5 =$ minus Peskin's as expected if we start from the same Minkowski space basis, but we have flipped the sign of γ^1 and γ^3! This is ok, since two minus signs give a plus.

Now look at the gauge part. In Minkowski space

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \theta\frac{g^2}{16\pi^2}F^{\mu\nu}\tilde{F}_{\mu\nu} && (18) \\ S &= \int dt \int d^3x\,\mathcal{L} && (19) \\ &= -\int dt \int d^3x \left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \theta\frac{g^2}{16\pi^2}F^{\mu\nu}\tilde{F}_{\mu\nu}\right) && (20) \end{aligned}$$

Analytically continue to Euclidean space,

$$\begin{aligned} t &\to -i\tau && (21) \\ F^{0i} &\to iF^{4i} && (22) \\ F_{0i} &\to -iF^{4i} && (23) \\ F^{ij} &\to -F^{ij} && (24) \\ F_{ij} &\to -F^{ij} && (25) \end{aligned}$$

The continuation of the field strength term can be worked out from

$$\begin{aligned} F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu && (26) \\ A^0 &\to -iA^4 && (27) \\ \tilde{F}_{\mu\nu} &= \epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} && (28) \end{aligned}$$

and similarly for the (contra)covariant (dual)field strength. A^μ is the four-vector potential and $\epsilon_{0123} = +1$. Then

$$\begin{aligned}
\exp iS &\rightarrow \exp\left\{i\int -id\tau \int d^3x \left(-\frac{1}{4}F^{\mu\nu}F^{\mu\nu} - i\theta\frac{g^2}{16\pi^2}F^{\mu\nu}\tilde{F}^{\mu\nu}\right)\right\} &(29)\\
&= \exp\left\{-\int d\tau \int d^3x \left(\frac{1}{4}F^{\mu\nu}F^{\mu\nu} + i\theta\frac{g^2}{16\pi^2}F^{\mu\nu}\tilde{F}^{\mu\nu}\right)\right\} &(30)\\
&= \exp\{-S_E\}. &(31)
\end{aligned}$$

The $F^{\mu\nu}F_{\mu\nu}$ term is easy- it does not flip sign. The θ term is a bit more work. It picks up an i since each contribution has one and only one factor of F^{0i} or F^{i0}. To get the sign right, it is easiest to just write out the whole term,

$$\begin{aligned}
F^{\mu\nu}\tilde{F}_{\mu\nu} &= 2\sum_i F^{0i}\tilde{F}_{0i} + 2\sum_{i,j>i} F^{ij}\tilde{F}_{ij} &(32)\\
\sum_{i\neq 0} F^{0i}\tilde{F}_{0i} &= F^{01}\tilde{F}_{01} + F^{02}\tilde{F}_{02} + F^{03}\tilde{F}_{03} &(33)\\
&= \frac{1}{2}(F^{01}(\epsilon_{0123}F^{23} + \epsilon_{0132}F^{32}) + F^{02}(\epsilon_{0213}F^{13} + \epsilon_{0231}F^{31}) + &(34)\\
&\quad F^{03}(\epsilon_{0321}F^{21} + \epsilon_{0312}F^{12})) &(35)\\
&= (F^{01}F^{23} - F^{02}F^{13} + F^{03}F^{12}) &(36)
\end{aligned}$$

and on analytic continuation

$$\begin{aligned}
\sum_{i\neq 0} F^{0i}\tilde{F}_{0i} &\rightarrow (iF^{41}(-1)F^{23} - iF^{42}(-1)F^{13} + iF^{43}(-1)F^{12}) &(37)\\
&= i(F^{14}F^{23} - F^{24}F^{13} + F^{34}F^{12}) &(38)\\
&= \frac{i}{2}\sum_{i\neq 4} F^{i4}\epsilon_{i4kl}F^{kl} &(39)\\
&= i\sum_{i\neq 4} F^{i4}\tilde{F}^{i4} &(40)
\end{aligned}$$

where $\epsilon_{1234} \equiv +1$.

$$\begin{aligned}
\sum_{i,j>i} F^{ij}\tilde{F}_{ij} &= F^{12}\tilde{F}_{12} + F^{13}\tilde{F}_{13} + F^{23}\tilde{F}_{23} &(41)\\
&= \frac{1}{2}\big(F^{12}(\epsilon_{1203}F^{03} + \epsilon_{1230}F^{30}) + F^{13}(\epsilon_{1302}F^{02} + \epsilon_{1320}F^{20}) &(42)\\
&\quad +F^{23}(\epsilon_{2301}F^{01} + \epsilon_{2310}F^{10})\big) &(43)\\
&= \big(F^{12}F^{03} - F^{13}F^{02} + F^{23}F^{01}\big) &(44)
\end{aligned}$$

On analytic continuation we have

$$\begin{aligned}
\big(F^{12}F^{03} - F^{13}F^{02} + F^{23}F^{01}\big) &\rightarrow \big(-F^{12}(i)F^{43} - -F^{13}(i)F^{42} + -F^{23}(i)F^{41}\big) &(45)\\
&= i\left(F^{12}F^{34} - F^{13}F^{24} + F^{23}F^{14}\right) &(46)\\
&= iF^{ij}\epsilon_{ij\alpha\beta}F^{\alpha\beta} &(47)\\
&= iF^{ij}\tilde{F}^{ij}. &(48)
\end{aligned}$$

Thus

$$\begin{aligned}
\exp iS &= \exp -i\theta \int dt \int d^3x F^{\mu\nu} \tilde{F}_{\mu\nu} && (49)\\
&\to \exp -i\theta \int -id\tau \int d^3x i F^{\mu\nu} \tilde{F}^{\mu\nu} && (50)\\
&= \exp i^3(-1)^2\theta \int d\tau \int d^3x F^{\mu\nu} \tilde{F}^{\mu\nu} && (51)\\
&= \exp -i\theta \int d\tau \int d^3x F^{\mu\nu} \tilde{F}^{\mu\nu}. && (52)
\end{aligned}$$

So the θ term does not change sign under analytic continuation.